The Travelling Wave

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1 Introduction

This article provides traceability between the equations of transmission line theory [1] and the circuit models used to simulate the transient responses of electrical systems [2].

The equations which define the behaviour of transient signals along the length of a twin-conductor transmission line are derived. The derivation and equations can be found in any book on Electromagnetic Theory. This article simply condenses the reasoning into a few pages using mathematics which is understandable to any student of Electrical Engineering.

The starting point is the definition of the circuit parameters in terms of distributed components. Then the circuit equations are derived for a minute section of the line.
Since current flows in a loop, along one conductor and back via the other, all the electrical parameters are defined as loop parameters. That is, they include the properties of both conductors. This avoids the need to invoke the concept of the equipotential conductor; and enables a better understanding to be reached of the mechanisms involved.

It is assumed that an alternating voltage is applied to the input terminals. To maintain a clear distinction between voltages and currents analysed in the frequency domain and those of the time domain, a distinctive font and colour is used for phasors. This distinction makes it possible to separate parameters which are functions of frequency from those which are functions of time and distance, and hence to identify the relationships of the travelling wave.

Expressions for the voltage developed along the conductors due to inductive effects and the voltage between the conductors due to capacitive effects are formulated. This leads to a pair of second order differential equations. A solution is identified, and it is shown that the current and voltage propagate along the line in the form of a travelling wave. Equations for the characteristic impedance and the propagation velocity are derived.

By limiting the analysis to the forward propagation of the wavefront, the mathematics is kept as simple as possible. By ignoring the effect of a backward-flowing wave, it is possible to show that the ratio of the rate of change of voltage with distance along the line to the rate of change of current with distance is also equal to the characteristic impedance. This means that if a voltage step is applied to the input terminals, then the current delivered to the line also undergoes a step change.

2 Characteristic Impedance

Figure 1 illustrates how voltage and current can vary along the length of a transmission line. For the purpose of deriving the transmission line equations, the following definitions apply:

R = loop resistance per unit length: Ω / m
L = loop inductance per unit length: H/m
C = loop capacitance per unit length: F/m
G = loop conductance per unit length: S/m

The parameters $\mathcal{V}$ and $\mathcal{I}$ are phasors, that is, functions of frequency. They are related to the peak voltage V and peak current I by:

$$\mathcal{V} = V \cdot e^{j \cdot \omega t} \tag{1}$$

$$\mathcal{I} = I \cdot e^{j \cdot \omega t} \tag{2}$$

where

$$\omega = 2 \cdot \pi \cdot f \tag{3}$$

and f is the frequency of the signal.

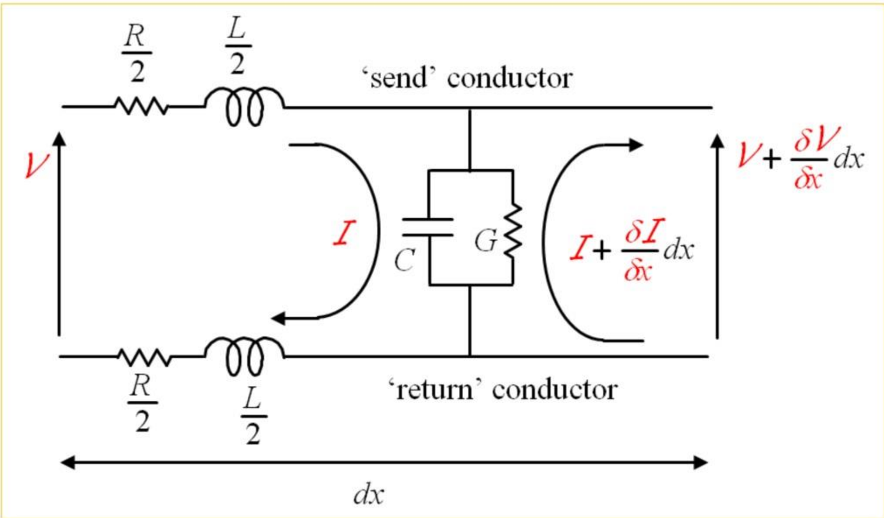

Fig. 1 Circuit model of one section of twin-conductor transmission line.

Voltage and current are also functions of x. For the finite section dx:

$$(R + j \cdot \omega \cdot L) \cdot dx \cdot \mathcal{I} = \mathcal{V} - \left(\mathcal{V} + \frac{\delta \mathcal{V}}{\delta x} \cdot dx \right) = -\frac{\delta \mathcal{V}}{\delta x} \cdot dx \tag{4}$$

$$(G + j \cdot \omega \cdot C) \cdot dx \cdot \mathcal{V} = \mathcal{I} - \left(\mathcal{I} + \frac{\delta \mathcal{I}}{\delta x} \cdot dx \right) = -\frac{\delta \mathcal{I}}{\delta x} \cdot dx \tag{5}$$

Hence

$$\frac{\delta V}{\delta x} = -(R + j \cdot \omega \cdot L) \cdot I \qquad (6)$$

$$\frac{\delta I}{\delta x} = -(G + j \cdot \omega \cdot C) \cdot V \qquad (7)$$

This removes the parameter *dx* from the analysis, temporarily. Differentiating (6) gives

$$\frac{\delta^2 V}{\delta x^2} = -(R + j \cdot \omega \cdot L) \cdot \frac{\delta I}{\delta x} = (R + j \cdot \omega \cdot L) \cdot (G + j \cdot \omega \cdot C) \cdot V \qquad (8)$$

Differentiating (7) gives

$$\frac{\delta^2 I}{\delta x^2} = -(G + j \cdot \omega \cdot C) \cdot \frac{\delta V}{\delta x} = (G + j \cdot \omega \cdot C) \cdot (R + j \cdot \omega \cdot L) \cdot I \qquad (9)$$

Substituting γ^2 for $(R + j \cdot \omega \cdot L) \cdot (G + j \cdot \omega \cdot C)$ in (8):

$$\frac{\delta^2 V}{\delta x^2} = \gamma^2 \cdot V \qquad (10)$$

That is

$$\gamma = \sqrt{(R + j \cdot \omega \cdot L) \cdot (G + j \cdot \omega \cdot C)} \qquad (11)$$

Equation (10) can be derived from:

$$V = Vn \cdot e^{-\gamma \cdot x} \qquad (12)$$

where Vn is the source voltage at the near end of the line.

$$Vn = Vn \cdot e^{j \cdot \omega t} \qquad (13)$$

and *Vn* is the peak amplitude of Vn. Differentiating (12):

$$\frac{\delta V}{\delta x} = -\gamma \cdot Vn \cdot e^{-\gamma \cdot x} \qquad (14)$$

Differentiating a second time results in equation (10).

$$\frac{\delta^2 V}{\delta x^2} = \gamma^2 \cdot Vn \cdot e^{-\gamma \cdot x} = \gamma^2 \cdot V$$

This confirms that equation (10) is valid.

Re-arranging equation (6):

$$I = -\frac{1}{R + j \cdot \omega \cdot L} \cdot \frac{\delta V}{\delta x}$$

Using (14) to substitute for $\frac{\delta V}{\delta x}$

$$I = \frac{\gamma}{R + j \cdot \omega \cdot L} \cdot Vn \cdot e^{-\gamma \cdot x} \quad (15)$$

Using (12) to substitute for $Vn \cdot e^{-\gamma \cdot x}$

$$I = \frac{\gamma}{R + j \cdot \omega \cdot L} \cdot V \quad (16)$$

Using (11) to substitute for γ

$$I = \sqrt{\frac{G + j \cdot \omega \cdot C}{R + j \cdot \omega \cdot L}} \cdot V \quad (17)$$

That is

$$I = \frac{V}{Zo} \quad (18)$$

where *Zo* is defined as the characteristic impedance:

$$Zo = \sqrt{\frac{R + j \cdot \omega \cdot L}{G + j \cdot \omega \cdot C}} \quad (19)$$

3 Velocity as a function of frequency

Since it is defined in terms of complex numbers, γ is a complex number. So it can be defined as:

$$\gamma = \alpha + j \cdot \beta \quad (20)$$

γ is described as the propagation constant, β as the wavelength factor and α as the attenuation factor. Using (20) to substitute for γ in (12)

$$V = Vn \cdot e^{-(\alpha + j \cdot \beta) \cdot x}$$

The voltage *V* at any instant *t* at the location *x* can be calculated by invoking the relationship defined by equation (1)

$$V = Vn \cdot e^{j \cdot \omega t} \cdot e^{-(\alpha + j \cdot \beta) \cdot x}$$

Re-arranging

$$V = Vn \cdot e^{-\alpha \cdot x} \cdot e^{j\omega \cdot \left(t - \frac{\beta \cdot x}{\omega}\right)} \tag{21}$$

This is an equation of the form

$$V = Vn \cdot e^{-\alpha \cdot x} \cdot e^{j\omega \cdot \left(t - \frac{x}{v}\right)} \tag{22}$$

where *v* is defined as:

$$v = \frac{\omega}{\beta} \tag{23}$$

4 The travelling wave

If, in equation (22), the term $\omega \cdot \left(t - \frac{x}{v}\right)$ is treated as a constant θ :

$$\left(t - \frac{x}{v}\right) = \frac{\theta}{\omega}$$

giving

$$x = v \cdot \left(t - \frac{\theta}{\omega}\right)$$

$$\frac{dx}{dt} = v$$

Since θ is constant and $\alpha = 0$ then $e^{j\theta}$ is constant and $e^{-\alpha \cdot x} = 1$. Under these conditions, *V* of equation (22) is constant. For this to happen, *x* must be increasing at the rate *v* as *t* increases. The relationship is similar to that of a surfer riding a wave. So long as the surfer maintains a fixed position on the wave, an observer sees the board and occupant travelling rapidly towards the shore.

Substituting for ω in (23)

$$v = \frac{2 \cdot \pi \cdot f}{\beta}$$

If τ is the time taken for a complete cycle

$$v = \frac{2 \cdot \pi}{\beta} \cdot \frac{1}{\tau}$$

If λ is the distance travelled in time τ

$$v = \frac{\lambda}{\tau} = \frac{2 \cdot \pi}{\beta} \cdot \frac{1}{\tau}$$

Giving

$$\lambda = \frac{2 \cdot \pi}{\beta} \tag{24}$$

5 Lossless line

It the line is lossless, then $R = 0$, $G = 0$, and $\alpha = 0$. So, from (11)

$$j \cdot \beta = \sqrt{(j \cdot \omega \cdot L) \cdot (j \cdot \omega \cdot C)} = j \cdot \omega \cdot \sqrt{L \cdot C}$$

So

$$\beta = \omega \cdot \sqrt{L \cdot C} \tag{25}$$

and, from (23)

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{L \cdot C}} \tag{26}$$

Also, from (19)

$$Ro = \sqrt{\frac{L}{C}} \tag{27}$$

Since the ratio of inductance to capacitance is a real number, then, for a lossless line, the impedance *Zo* can have no imaginary component. So it is defined to be a pure resistance in equation (27).

From (18)

$$\mathcal{V} = Ro \cdot \mathcal{I}$$

Hence

$$V \cdot e^{j \cdot \omega t} = Ro \cdot I \cdot e^{j \cdot \omega t}$$

$$V = Ro \cdot I \tag{28}$$

This means, that, for a lossless line, The load presented to the input terminals is constant. It does not matter what circuitry is present at the far end.

6 Rates of change

It has been shown that, if a sine wave signal is applied to the input terminals of a transmission line, then that signal will propagate along that line at a velocity given by equation (26). A further characteristic of the line can be identified:

Differentiating (15) with respect to *x*:

$$\frac{\delta I}{\delta x} = \frac{-\gamma^2}{R + j\cdot\omega\cdot L}\cdot Vn\cdot e^{-\gamma\cdot x} \tag{29}$$

Dividing (14) by (29):

$$\frac{\delta V}{\delta x}\Big/\frac{\delta I}{\delta x} = -\gamma\cdot Vn\cdot e^{-\gamma\cdot x}\cdot\frac{R + j\cdot\omega\cdot L}{-\gamma^2\cdot Vn\cdot e^{-\gamma\cdot x}}$$

giving

$$\frac{\delta V}{\delta x}\Big/\frac{\delta I}{\delta x} = \frac{R + j\cdot\omega\cdot L}{\gamma}$$

Using (11) to substitute for γ

$$\frac{\delta V}{\delta x} = \sqrt{\frac{R + j\cdot\omega\cdot L}{G + j\cdot\omega\cdot C}}\cdot\frac{\delta I}{\delta x}$$

Invoking (19)

$$\frac{\delta V}{\delta x} = Zo\cdot\frac{\delta I}{\delta x} \tag{30}$$

For a lossless line: $$\frac{\delta V}{\delta x} = Ro\cdot\frac{\delta I}{\delta x}$$

and $$\frac{dV}{dx}\cdot e^{j\cdot\omega\cdot t} = Ro\cdot\frac{dI}{dx}\cdot e^{j\cdot\omega\cdot t}$$

giving $$\frac{dV}{dx} = Ro\cdot\frac{dI}{dx} \tag{31}$$

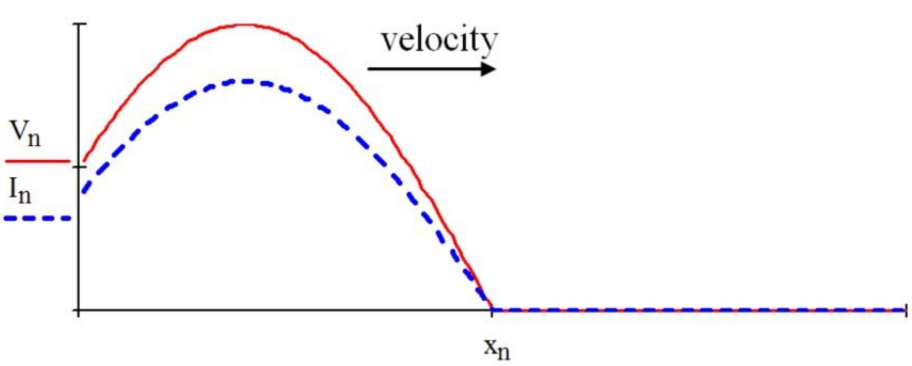

Figure 2 Propagation of a sine wave along transmission line

That is, as well as the current being proportional to the voltage at any point x, the rate of change of current with distance is also proportional to the rate of change of voltage with distance. Figure 2 illustrates this. The significant feature of the relationship is that there is no phase difference between the two parameters, whatever the frequency.

For a lossless line, the step between equation (30) and (31) causes the frequency parameters ω and f to drop out of the equations.

7 Response to a step voltage

If a step voltage is applied, the resulting change in current is also a step change. At the first instant after a change is applied at the near end of a transmission line the voltage and current waveforms propagate along the line at a velocity defined by equation (26). Figure 3 illustrates this.

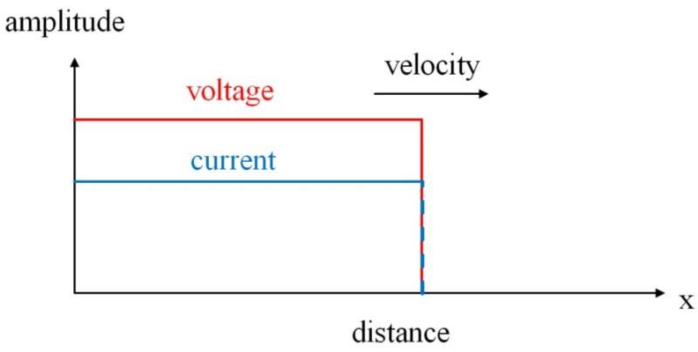

Figure 3 Propagation of a step voltage and a step current along a transmission line.

The key feature about the relationship depicted in Figure 3 is the fact that an instantaneous change in voltage is accompanied by an instantaneous change in current. The relationship between current and voltage is defined by equation (28).

With a step waveform, the initial step is followed by a steady flow of current I into the line. Behind the wavefront, the voltage is also constant.

8 Propagation velocity.

If it is assumed that the transmission line is configured as a twin-conductor cable, then the value of the inductance per unit length will be

$$L = \frac{\mu_o \cdot \mu_r}{\pi} \cdot \ln\left(\frac{sep}{rad}\right) \qquad (32)$$

where *rad* is the radius of each conductor, *sep* is the separation between the centres, μ_o is the permeability of free space and μ_r is the relative permeability of the conductor.

The capacitance per unit length will be:

$$C = \frac{\pi \cdot \varepsilon_o \cdot \varepsilon_r}{\ln\left(\frac{sep}{rad}\right)} \tag{33}$$

where ε_o is the permittivity of free space ε_r is the average relative permittivity of the insulation between the conductors. The velocity of propagation is

$$v = \frac{1}{\sqrt{L \cdot C}} = \frac{1}{\sqrt{\mu_o \cdot \mu_r \cdot \varepsilon_o \cdot \varepsilon_r}} \tag{34}$$

The velocity of light in a vacuum is

$$c = \frac{1}{\sqrt{\mu_o \cdot \varepsilon_o}} \tag{35}$$

Since the velocity of light is a constant, the parameters μ_r and ε_r effectively define the actual velocity of propagation. The value of μ_r is the property of the conductors. It can be assumed that, for copper, $\mu_r = 1$. The value of ε_r is the property of the insulating material. Since more than one type of material is involved in insulating one conductor from another, it is difficult to assign a value to the relative permittivity, even when data is available on the properties of each type of insulation.

There is another factor to consider. The transmission line model of Figure 1 depicts two conductors, where current flows along the send conductor, across the gap between the conductors, and back along the return conductor. It also indicates the current flows back along the return conductor and then across the gap into the send conductor. Time will be spent in criss-crossing the gap between the conductors. This will delay the arrival of the leading edge at the far end.

However, by measuring the time it takes for the leading edge to propagate along a sample length of cable, the velocity of propagation can be measured and a value assigned to ε_r. This value can be used when analysing the characteristics of any signal carried by that type of cable.

9 Propagation time

If the length of the line is *len*, then the time *T* for the leading edge to propagate from the near end of the line to the far end is

$$T = \frac{len}{v} \tag{36}$$

Since the current *I* is constant, the charge delivered to the line at time *T* will be

$$Q = I \cdot T \tag{37}$$

For the length *len*, the actual capacitance *Ca* between the conductors is

$$Ca = C \cdot len \tag{38}$$

and the voltage between the conductors is

$$V = \frac{Q}{Ca} \tag{39}$$

The fact that charges have been delivered to the entire length of the line during time *T* means that, as far as electromagnetic theory is concerned, charge propagates at a speed comparable to that of light.

10 The return conductor

Figure 1 illustrates the fact that current flows along the send conductor, across the gap between the two conductors, back along the send conductor and then back across the gap into the send conductor.

The reason that current flows back into the send conductor is due to the existence of the voltage developed along the return conductor. If the voltage along the return conductor did not exist, then current would flow out of the send conductor and disappear into the environment.

This leads to two conclusions.

> Two conductors are necessary to steer the electromagnetic energy along the path defined by the routing of the cable.
>
> In any functioning electronic system, there is no such thing as an equipotential conductor.

11 Conclusion

It has been shown that electrical signals propagate along a twin-conductor transmission line in the form of a travelling wave. Formulae for the characteristic impedance and the propagation velocity have been defined in terms of distributed parameters; ohm/metre, Henry/metre, Farad/metre, and mho/metre.

When a step voltage is applied to the input terminals, the waveform of the current also undergoes a step change. There is no delay between the application of a voltage and the creation of a current. An instantaneous change in voltage is accompanied by an instantaneous change in current. For this to happen, current and voltage must be manifestations of the same mechanism; the flow of charge.

It has been shown that the velocity of propagation of the electromagnetic energy is slowed down by the fact that current flows between the conductors as well as along the conductors. The phenomenon can be catered for in any circuit model by assigning a value to the relative permittivity.

It has also been shown that two conductors are necessary to ensure the efficient transmission of energy from source to load, and that there is no such thing as an equipotential conductor.

12 References

[1] Skitek, G. G., Marshal, S. V.: Transmission lines', in 'Electromagnetic concepts and applications' (Prentice Hall, Inc., 1982). Pages 372-374

[2] Transients. www.designemc.info/15Transients.pdf